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A criterion for convergence of  
weak greedy algorithms

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# A criterion for convergence of Weak Greedy Algorithms<sup>1</sup>

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## 1. INTRODUCTION

This paper completes the investigation of necessary and sufficient conditions on the "weakness" sequence  $\tau := \{t_k\}_{k=1}^\infty$  for convergence of Weak Greedy Algorithm for all dictionaries  $\mathcal{D}$  and each function (vector)  $f$  in Hilbert space  $H$ . This paper is a follow up to the papers [T] and [LT]. The Weak Greedy Algorithms (WGA) were introduced in [T]. The paper [T] contains also historical remarks and some motivation of studying greedy and weak greedy algorithms. We will not repeat historical remarks from [T] here and refer the reader to [T] for prehistory of WGA. We discuss here results on WGA in detail.

We remind first some notations and definitions from the theory of greedy algorithms. Let  $H$  be a real Hilbert space with an inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\|x\| := \langle x, x \rangle^{1/2}$ . We say a set  $\mathcal{D}$  of functions (elements) from  $H$  is a dictionary if each  $g \in \mathcal{D}$  has norm one ( $\|g\| = 1$ ) and  $\overline{\text{span}}\mathcal{D} = H$ . We give now the definition of WGA (see [T]). Let a weakness sequence  $\tau = \{t_k\}_{k=1}^\infty$ ,  $0 \leq t_k \leq 1$ , be given.

**Weak Greedy Algorithm.** *We define  $f_0^\tau := f$ . Then for each  $m \geq 1$ , we inductively define:*

1).  $\varphi_m^\tau \in \mathcal{D}$  is any satisfying

$$|\langle f_{m-1}^\tau, \varphi_m^\tau \rangle| \geq t_m \sup_{g \in \mathcal{D}} |\langle f_{m-1}^\tau, g \rangle|;$$

2).

$$f_m^\tau := f_{m-1}^\tau - \langle f_{m-1}^\tau, \varphi_m^\tau \rangle \varphi_m^\tau;$$

3).

$$G_m^\tau(f, \mathcal{D}) := \sum_{j=1}^m \langle f_{j-1}^\tau, \varphi_j^\tau \rangle \varphi_j^\tau.$$

In the case  $t_k = 1$ ,  $k = 1, 2, \dots$ , we call WGA by Pure Greedy Algorithm (PGA). The convergence of PGA and WGA with  $t_k = t$ ,  $0 < t < 1$ , was established in [J] and [RW]. The first sufficient condition on  $\tau$  which includes sequences with  $\lim_{k \rightarrow \infty} t_k = 0$  was obtained in [T].

**Theorem A.** *Assume*

$$(1.1) \quad \sum_{k=1}^{\infty} \frac{t_k}{k} = \infty.$$

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Then for any dictionary  $\mathcal{D}$  and any  $f \in H$  we have

$$\lim_{m \rightarrow \infty} \|f - G_m^\tau(f, \mathcal{D})\| = 0.$$

In [T] we reduced the proof of convergence of WGA with weakness sequence  $\tau$  to some properties of  $l_2$ -sequences with regard to  $\tau$ . Theorem A was derived from the following two statements proved in [T].

**Proposition 1.1.** *Let  $\tau$  be such that for any  $\{a_j\}_{j=1}^\infty \in l_2$ ,  $a_j \geq 0$ ,  $j = 1, 2, \dots$  we have*

$$\liminf_{n \rightarrow \infty} a_n \sum_{j=1}^n a_j / t_n = 0.$$

*Then for any  $H$ ,  $\mathcal{D}$ , and  $f \in H$  we have*

$$\lim_{m \rightarrow \infty} \|f_m^\tau\| = 0.$$

**Proposition 1.2 (Lemma 2.3,[T]).** *If  $\tau$  satisfies the condition (1.1) then  $\tau$  satisfies the assumption of Proposition 1.1.*

The following simple necessary condition

$$\sum_{k=1}^{\infty} t_k^2 = \infty$$

was mentioned in [T]. The first nontrivial necessary conditions were obtained in [LT]. We proved in [LT] the following theorem.

**Theorem B.** *In the class of monotone sequences  $\tau = \{t_k\}_{k=1}^\infty$ ,  $1 \geq t_1 \geq t_2 \geq \dots \geq 0$ , the condition (1.1) is necessary and sufficient for convergence of Weak Greedy Algorithm for each  $f$  and all Hilbert spaces  $H$  and dictionaries  $\mathcal{D}$ .*

The proof of this theorem is based on a special procedure which we called Equalizer. The generalization of that procedure plays an important role in this paper also (see S.3). In [LT] we gave an example of a class of sequences  $\tau$  for which the condition (1.1) is not a necessary condition for convergence. We also proved in [LT] a theorem which covers Theorem A.

**Theorem C.** *Assume*

$$\sum_{s=0}^{\infty} (2^{-s} \sum_{k=2^s}^{2^{s+1}-1} t_k^2)^{1/2} = \infty.$$

*Then for any dictionary  $\mathcal{D}$  and any  $f \in H$  we have*

$$\lim_{m \rightarrow \infty} \|f - G_m^\tau(f, \mathcal{D})\| = 0.$$

We prove in this paper a criterion on  $\tau$  for convergence of WGA. Let us introduce some notation.

We define by  $\mathcal{V}$  the class of sequences  $x = \{x_k\}_{k=1}^\infty$ ,  $x_k \geq 0$ ,  $k = 1, 2, \dots$ , with the following property: there exists a sequence  $0 = q_0 < q_1 < \dots$  such that

$$(1.2) \quad \sum_{s=1}^{\infty} \frac{2^s}{\Delta q_s} < \infty;$$

and

$$(1.3) \quad \sum_{s=1}^{\infty} 2^{-s} \sum_{k=1}^{q_s} x_k^2 < \infty,$$

where  $\Delta q_s := q_s - q_{s-1}$ .

**Remark 1.1.** It is clear from this definition that if  $x \in \mathcal{V}$  and for some  $N$  and  $c$  we have  $0 \leq y_k \leq cx_k$ ,  $k \geq N$ , then  $y \in \mathcal{V}$ .

**Theorem 1.1.** *The condition  $\tau \notin \mathcal{V}$  is necessary and sufficient for convergence of Weak Greedy Algorithm with weakness sequence  $\tau$  for each  $f$  and all Hilbert spaces  $H$  and dictionaries  $\mathcal{D}$ .*

Sufficient part is proved in Section 2 and necessary part is proved in Section 3.

## 2. PROOF OF CONVERGENCE

We begin this section with the following lemma.

**Lemma 2.1.** *Let  $\{a_j\}_{j=1}^{\infty} \in l_2$ ,  $a_j \geq 0$ ,  $j = 1, 2, \dots$ . Then  $\{a_n \sum_{j=1}^n a_j\}_{n=1}^{\infty} \in \mathcal{V}$ .*

*Proof.* Assume  $\{a_j\}_{j=1}^{\infty}$  contains infinitely many nonzero terms (if not the statement is trivial). Denote  $y_n := a_n \sum_{j=1}^n a_j$  and define  $q_s := q_s(y)$  inductively:  $q_0 := 0$  and for  $q_0, \dots, q_{s-1}$  defined we choose  $q_s$  as the smallest  $q$  such that

$$(2.1) \quad (q - q_{s-1}) \sum_{n=q_{s-1}+1}^q y_n^2 \geq 2^{2s}.$$

Denote  $Q_s := (q_{s-1}, q_s]$ . Then (2.1) implies

$$\frac{2^s}{\Delta q_s} \leq 2^{-s} \sum_{n \in Q_s} y_n^2 \leq 2^{-s} \sum_{n=1}^{q_s} y_n^2.$$

Thus it is sufficient to check only (1.3)

$$\sum_s 2^{-s} \sum_{n=1}^{q_s} y_n^2 < \infty.$$

From the definition of  $q_s$  we have

$$(2.2) \quad \sum_{n=q_{s-1}+1}^{q_s-1} y_n \leq (\Delta q_s - 1)^{1/2} \left( \sum_{n=q_{s-1}+1}^{q_s-1} y_n^2 \right)^{1/2} < 2^s.$$

Next for any  $N \leq M$  we have

$$\sum_{n=N}^M a_n \sum_{j=1}^n a_j \geq \sum_{N \leq j \leq n \leq M} a_n a_j =$$

$$(2.3) \quad = 1/2 \left( \sum_{j=N}^M a_j^2 + (\sum_{j=N}^M a_j)^2 \right) \geq (\sum_{j=N}^M a_j)^2 / 2.$$

Combining (2.2) and (2.3) we get

$$\sum_{j \in Q_s} a_j = \sum_{j=q_{s-1}+1}^{q_s-1} a_j + a_{q_s} \leq 2^{(s+1)/2} + \|a\|_\infty.$$

This implies

$$(2.4) \quad \sum_{j=1}^{q_s} a_j \leq C(a) 2^{s/2}.$$

We have now

$$\begin{aligned} \sum_s 2^{-s} \sum_{n=1}^{q_s} y_n^2 &= \sum_s 2^{-s} \sum_{v=1}^s \sum_{n \in Q_v} y_n^2 \leq 2 \sum_s 2^{-s} \sum_{n \in Q_s} y_n^2 \leq \\ 2 \sum_s 2^{-s} (\sum_{j=1}^{q_s} a_j)^2 \sum_{n \in Q_s} a_n^2 &\leq C(a) \sum_n a_n^2 < \infty. \end{aligned}$$

Lemma 2.1 is proved now.

**Theorem 2.1.** *The following two conditions are equivalent*

$$(C.1) \quad \tau \notin \mathcal{V},$$

$$(C.2) \quad \forall \{a_j\}_{j=1}^\infty \in l_2, \quad a_j \geq 0, \quad \liminf_{n \rightarrow \infty} a_n \sum_{j=1}^n a_j / t_n = 0.$$

*Proof.* We prove first that (C.1)  $\Rightarrow$  (C.2). Assume (C.2) is not satisfied:  $\exists \{a_j\}_{j=1}^\infty \in l_2, a_j \geq 0$ , such that

$$(2.5) \quad \liminf_{n \rightarrow \infty} a_n \sum_{j=1}^n a_j / t_n > 0.$$

Relation (2.5) implies that for some  $N$  and  $c > 0$  we have for  $n \geq N$  that

$$a_n \sum_{j=1}^n a_j / t_n \geq c$$

or

$$t_n \leq C a_n \sum_{j=1}^n a_j.$$

This inequality, Lemma 2.1, and Remark 1.1 imply that  $\tau \in \mathcal{V}$ . The first implication is proved now.

We proceed to the second implication (C.2)  $\Rightarrow$  (C.1). Let  $\tau \in \mathcal{V}$ . We construct a sequence  $\{a_j\}_{j=1}^{\infty} \in l_2$  such that for all  $n$

$$t_n \leq C a_n \sum_{j=1}^n a_j$$

with some  $C$ . This will imply that (C.2) is not satisfied. Let  $\{q_s\} := \{q_s(\tau)\}$  be a sequence from the definition of  $\mathcal{V}$ . We define a sequence  $\{a_j\}_{j=1}^{\infty}$  as follows. For  $n \in Q_s$  we set

$$a_n := t_n 2^{-s/2} + 2^{s/2}/\Delta q_s.$$

Then

$$a_n^2 \leq 2(t_n^2 2^{-s} + 2^s (\Delta q_s)^{-2})$$

and

$$\sum_n a_n^2 \leq 2 \sum_s 2^{-s} \sum_{n \in Q_s} t_n^2 + 2 \sum_s \frac{2^s}{\Delta q_s} < \infty.$$

Next,

$$\sum_{n \in Q_s} a_n \geq 2^{s/2}.$$

Thus for  $n \in Q_s$  we have

$$a_n \sum_{j=1}^n a_j \geq a_n \sum_{j \in Q_{s-1}} a_j \geq t_n 2^{-1/2}$$

and

$$t_n \leq \sqrt{2} a_n \sum_{j=1}^n a_j$$

for all  $n$ .

Theorem 2.1 is proved now.

The sufficient part of Theorem 1.1 follows from Theorem 2.1 and Proposition 1.1.

### 3. CONSTRUCTION OF A COUNTEREXAMPLE

The following procedure which is the generalization of Equalizer from [LT] plays an important role in the construction. Let  $H$  be a Hilbert space with an orthonormal basis  $\{e_j\}_{j=1}^{\infty}$ . We take two elements  $e_i, e_j$ ,  $i \neq j$ , and define the following procedure.

**Equalizer with schedule**  $\gamma := \{\gamma_k\}$ . Let  $\gamma_k \leq 1/5$ ,  $f_0 := e_i$ . Define:

$$(3.1) \quad g_1 := \alpha_1 e_i - (1 - \alpha_1^2)^{1/2} e_j; \quad \alpha_1 = \gamma_1; \quad \langle f_0, g_1 \rangle = \gamma_1;$$

$$(3.2) \quad f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n; \quad g_n := \alpha_n e_i - (1 - \alpha_n^2)^{1/2} e_j;$$

$$\langle f_n, g_{n+1} \rangle = \gamma_{n+1}; \quad f_n = a_n e_i + b_n e_j.$$

We check

$$a_{n-1} - b_{n-1} \geq 3\sqrt{2}\gamma_n$$

to continue. If

$$a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n$$

then we take  $g_n := 2^{-1/2}(e_i - e_j)$  and

$$f_n := f_{n-1} - \langle f_{n-1}, g_n \rangle g_n,$$

and stop after this step. We call this step "the final step" and all other steps "regular steps". At each regular step  $l$  we have

$$a_l - b_l = a_{l-1} - b_{l-1} - \gamma_l(\alpha_l + (1 - \alpha_l^2)^{1/2}) \geq a_{l-1} - b_{l-1} - 2^{1/2}\gamma_l > 0.$$

After the final step we have

$$a_n = b_n.$$

At each regular step we have by definition that

$$\langle f_{l-1}, g_l \rangle = \gamma_l.$$

At the final step we have

$$\begin{aligned} \langle f_{n-1}, g_n \rangle &= 2^{-1/2}(a_{n-1} - b_{n-1}) \geq 2^{-1/2}(a_{n-2} - b_{n-2} - 2^{1/2}\gamma_{n-1}) \geq \\ &\geq 2^{-1/2}(2\sqrt{2}\gamma_{n-1}) = 2\gamma_{n-1}. \end{aligned}$$

Thus, if  $2\gamma_{n-1} \geq \gamma_n$  then the above described Equalizer is a WGA with weakness sequence  $\gamma_1, \dots, \gamma_n$ .

At regular step  $l$  we reduce the  $\|\cdot\|^2$  by  $\gamma_l^2$ . At the final step we reduce the  $\|\cdot\|^2$  by

$$\frac{1}{2}(a_{n-1} - b_{n-1})^2 < 9\gamma_n^2.$$

We also have

$$a_{n-1} - b_{n-1} < 3\sqrt{2}\gamma_n$$

and

$$a_{n-1} - b_{n-1} \geq 1 - \sqrt{2} \sum_{j=1}^{n-1} \gamma_j.$$

Thus,

$$\sqrt{2} \sum_{j=1}^{n-1} \gamma_j + 3\sqrt{2}\gamma_n > 1.$$

On the other hand

$$a_l - b_l \leq a_{l-1} - b_{l-1} - \gamma_l.$$

Therefore,

$$0 \leq a_{n-1} - b_{n-1} \leq 1 - \sum_{l=1}^{n-1} \gamma_l$$

and

$$(3.3) \quad \sum_{l=1}^{n-1} \gamma_l \leq 1.$$

In order to apply the above Equalizer we need to have the inequality  $2\gamma_{n-1} \geq \gamma_n$  satisfied. Let us use the following regularization procedure.

**Regularization.** For a given  $\tau = \{t_k\}_{k=1}^\infty$ ,  $\tau \in l_\infty$ , we define  $\tau^R := \{t_k^R\}_{k=1}^\infty$  with

$$t_k^R := \sum_{m=0}^{\infty} 2^{-m} t_{n+m}.$$

**Lemma 3.1.** *If  $\tau \in \mathcal{V} \cap l_\infty$  then  $\tau^R \in \mathcal{V} \cap l_\infty$ .*

*Proof.* Assumption  $\tau \in \mathcal{V}$  implies

$$(3.4) \quad \sum_s 2^{-s} \sum_{k=1}^{q_s} t_k^2 < \infty.$$

We will prove that

$$(3.5) \quad \sum_s 2^{-s} \sum_{k=1}^{q_s} (t_k^R)^2 < \infty$$

with the same  $q_s = q_s(\tau)$  as above. Thus (3.5) will imply  $\tau^R \in \mathcal{V}$ . Let us prove (3.5). We have for any  $N$

$$\begin{aligned} \sum_{k=1}^N (t_k^R)^2 &= \sum_{k=1}^N \left( \sum_{m=0}^\infty 2^{-m} t_{n+m} \right)^2 = \sum_{k=1}^N \sum_{m=0}^\infty \sum_{n=0}^\infty 2^{-m-n} t_{k+m} t_{k+n} = \\ &\sum_{m=0}^\infty \sum_{n=0}^\infty 2^{-m-n} \sum_{k=1}^N t_{k+m} t_{k+n} \leq \sum_{m=0}^\infty \sum_{n=0}^\infty 2^{-m-n} \left( \sum_{k=1}^N t_{k+m}^2 \right)^{1/2} \left( \sum_{k=1}^N t_{k+n}^2 \right)^{1/2} = \\ &\left( \sum_{m=0}^\infty 2^{-m} \left( \sum_{k=1}^N t_{k+m}^2 \right)^{1/2} \right)^2 \leq \left( \sum_{m=0}^\infty 2^{-m} \right) \left( \sum_{m=0}^\infty 2^{-m} \sum_{k=1}^N t_{k+m}^2 \right). \end{aligned}$$

Next,

$$\sum_{k=1}^N t_{k+m}^2 \leq \sum_{k=1}^N t_k^2 + m \|\tau\|_\infty^2$$

and

$$\sum_{m=0}^\infty 2^{-m} \left( \sum_{j=1}^N t_j^2 + m \|\tau\|_\infty^2 \right) \leq 2 \sum_{j=1}^N t_j^2 + C(\tau).$$

Therefore we got

$$\sum_{j=1}^N (t_j^R)^2 \leq 2 \sum_{j=1}^N t_j^2 + C(\tau)$$

and

$$\sum_s 2^{-s} \sum_{k=1}^{q_s} (t_k^R)^2 \leq 2 \sum_s 2^{-s} \sum_{k=1}^{q_s} t_k^2 + C(\tau) < \infty.$$

It is easy to see that  $\|\tau^R\|_\infty \leq 2 \|\tau\|_\infty$ .

Lemma 3.1 is proved now.

Thus for any  $\tau \in \mathcal{V} \cap l_\infty$  we have  $\tau^R \in \mathcal{V}$ ,  $\|\tau^R\|_\infty \leq 2\|\tau\|_\infty$ , and

$$2t_{n-1}^R \geq t_n^R, \quad n = 2, 3, \dots$$

Clearly, we also have for all  $n$

$$t_n \leq t_n^R.$$

One more restriction in the Equalizer is  $\gamma_n \leq 1/5$ . Define a new sequence  $\tau'$  by

$$t'_n := \min\{t_n^R, 1/5\}.$$

It is clear that  $\tau' \in \mathcal{V}$  and also satisfies

$$2t'_{n-1} \geq t'_n.$$

Let  $\{q_s\} := \{q_s(\tau')\}$  be the sequence for  $\tau'$  from the definition of  $\mathcal{V}$ :

$$\sum_s \frac{2^s}{\Delta q_s} < \infty, \quad \sum_s 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \infty.$$

Let  $\epsilon$  be a small number which we will specify later and  $s_0$  be such that

$$\sum_{s \geq s_0} \frac{2^s}{\Delta q_s} < \epsilon, \quad \sum_{s \geq s_0} 2^{-s} \sum_{n=1}^{q_s} (t'_n)^2 < \epsilon.$$

Consider the function

$$f_{s_0} := 2^{-s_0/2}(e_1 + \dots + e_{2^{s_0}}).$$

We have  $\|f_{s_0}\| = 1$ . Define

$$t''_k := \max\{t'_k, 2^{s_0+2}(\Delta q_{s_0})^{-1}\}.$$

We apply a mixture of Equalizer with schedule  $\{t''_k\}$  to vectors  $e_i$ ,  $i \leq 2^{s_0}$ , and the PGA to the corresponding residual of  $f_{s_0}$ . We do this in the following way. If  $t''_1 \geq 1/5$  we use PGA and throw away, say,  $2^{-s_0/2}e_{2^{s_0}}$ . If  $t''_1 < 1/5$  we start using the Equalizer with schedule  $\{t''_k\}$  to vectors  $e_1$  and  $e_{2^{s_0}+1}$ . If at some step  $t''_k \geq 1/5$  then we use PGA what means throwing away one term of the form  $2^{-s_0/2}e_j$ ,  $j \in [1, 2^{s_0}]$ . Applying the Equalizer to the very last term of the form  $2^{-s_0/2}e_m$  we may encounter with  $t''_k \geq 1/5$ . In such a case we apply PGA and stop. As a result we get

$$f_{s_0+1} := \sum_{k \in F_{s_0+1}} c_k^{s_0+1} e_k.$$

It is clear that for all  $k \in F_{s_0+1}$  we have

$$(c_k^{s_0+1})^2 \leq 2^{-s_0-1}$$

and also

$$|F_{s_0+1}| \leq 2^{s_0+1}.$$

Assume that  $\epsilon < 1/20$ . Then  $2^{s_0+2}(\Delta q_{s_0})^{-1} < 1/5$  and  $t''_k \geq 1/5$  is equivalent to  $t''_k = t'_k = 1/5$ . If  $t''_k < 1/5$  then  $t'_k < 1/5$  and  $t_k \leq t'_k$ . Therefore, at all Equalizer steps we have a WGA with weakness parameters  $\{t_k\}$ . If  $t''_k = 1/5$  we apply PGA what is a WGA with any  $t_k$  at this step. During this procedure which we call "working on  $s_0$ -level" we perform  $M_{s_0}^w$  steps of Equalizer and  $M_{s_0}^G$  steps of PGA. Let us estimate  $M_{s_0}^w$  and  $M_{s_0}^G$ . It is clear that  $M_{s_0}^G \leq 2^{s_0}$ . We have applied the Equalizer to terms of the form  $2^{-s_0/2}e_j$  at most  $2^{s_0}$  times. For each Equalizer application we have  $\sum \gamma_j \leq 2$  (see (3.3)). Thus denoting  $E(s_0) := \{k : t''_k < 1/5\}$  we get

$$\sum_{k \in E(s_0)} t''_k \leq 2^{s_0+1}.$$

On the other hand we have

$$\sum_{k \in E(s_0)} t''_k \geq M_{s_0}^w 2^{s_0+2}(\Delta q_{s_0})^{-1}$$

and

$$M_{s_0}^w \leq \Delta q_{s_0}/2.$$

Therefore,

$$N_{s_0} := M_{s_0}^w + M_{s_0}^G \leq \Delta q_{s_0}/2 + 2^{s_0} \leq \Delta q_{s_0}.$$

At each Equalizer step we reduced the  $\|\cdot\|^2$  by at most  $9(t''_k)^2 2^{-s_0}$  and at each PGA by at most  $25(t'_k)^2 2^{-s_0}$ . Thus the total reduction  $\delta_{s_0}$  for the  $s_0$ -level does not exceed

$$25(2^{-s_0}) \sum_{k=1}^{q_{s_0}} (t'_k)^2 + 9(2^{s_0+4})(\Delta q_{s_0})^{-1}.$$

We are on the  $(s_0 + 1)$ -level now and perform the similar procedure. We describe it for the general case of an  $s$ -level. Assume we have after  $N_{s-1} \leq q_{s-1}$  steps of our WGA the function

$$f_s = \sum_{k \in F_s} c_k^s e_k$$

with

$$(c_k^s)^2 \leq 2^{-s}, \quad |F_s| \leq 2^s.$$

Define now

$$t''_k := \max\{t'_k, 2^{s+2}(\Delta q_s)^{-1}\}, \quad k > N_{s-1}.$$

We pick  $c_k^s e_k$  with the biggest  $c_k^s$  out of  $\{c_k^s, k \in F_s\}$  and throw it away if  $t''_{N_{s-1}+1} = 1/5$  (we remind that assumption  $\epsilon < 1/20$  implies  $2^{s+2}(\Delta q_s)^{-1} < 1/5$ ) and apply the Equalizer with schedule  $\{t''_n\}$  otherwise. We continue to perform the above described procedure (the mixture of Equalizer and PGA steps) until we get

$$f_{s+1} = \sum_{k \in F_{s+1}} c_k^{s+1} e_k$$

with

$$(c_k^{s+1})^2 \leq 2^{-s-1}.$$

It is clear that then  $|F_{s+1}| \leq 2^{s+1}$ . Similarly to the above estimates of  $M_{s_0}^w$  and  $M_{s_0}^G$  we get

$$M_s^G \leq 2^s$$

and

$$M_s^w 2^{s+2} (\Delta q_s)^{-1} \leq \sum_{k \in E(s)} t_k'' \leq 2^{s+1}.$$

Thus

$$M_s^w + M_s^G \leq \Delta q_s / 2 + 2^s \leq \Delta q_s$$

and

$$N_s := N_{s-1} + M_s^w + M_s^G \leq q_s.$$

The total reduction  $\delta_s$  of the  $\|\cdot\|^2$  from working on the  $s$ -level does not exceed

$$25(2^{-s}) \sum_{k=1}^{q_s} (t'_k)^2 + 9(2^{s+4})(\Delta q_s)^{-1}.$$

We continue this process and get that the  $\|\cdot\|^2$  will be reduced by at most

$$\sum_{s=s_0}^{\infty} \delta_s \leq 25 \left( \sum_{s=s_0}^{\infty} 2^{-s} \sum_{k=1}^{q_s} (t'_k)^2 \right) + 144 \sum_{s=s_0}^{\infty} 2^s (\Delta q_s)^{-1} \leq 169\epsilon.$$

Choosing  $\epsilon$  small enough, say,  $\epsilon = 0.005$  we get divergent WGA with the weakness sequence  $\tau$ . This completes the construction of the counterexample.

## REFERENCES

- [J] L. Jones, *On a conjecture of Huber concerning the convergence of projection pursuit regression*, The Annals of Statistics **15** (1987), 880–882.
- [LT] E.D. Livshitz and V.N. Temlyakov, *On convergence of Weak Greedy Algorithms*, IMI-Preprint (2000), no. 13, 1–9.
- [RW] L. Rejtö and G.G. Walter, *Remarks on projection pursuit regression and density estimation*, Stochastic Analysis and Application **10** (1992), 213–222.
- [T] V.N. Temlyakov, *Weak Greedy Algorithms*, Advances in Computational Mathematics **12** (2000), 213–227.